

IDENTITIES FOR $\sin x$ THAT CAME FROM MEDICAL IMAGING

PETER KUCHMENT AND SERGEY LVIN

*Dedicated to the memory of Professor Leon Ehrenpreis,
a great mathematician and human being.*

ABSTRACT. The article describes interesting nonlinear differential identities satisfied by standard exponential and trigonometric functions, which appeared as byproducts of medical imaging research. They look like some kind of non-commutative binomial formulas. A brief description of the origin of these identities is provided, as well as their direct algebraic derivation. Relations with separate analyticity theorems in several complex variables and some open problems are also mentioned.

1. INTRODUCTION

Working quite some time ago on some problems of medical imaging [19, 20], the authors were surprised when their work produced a series of seemingly new (at least to them) identities for functions about which “everything is known,” such as standard exponentials and trigonometric and hyperbolic sines and cosines. It was a little bit depressing when our brave attempt to prove these identities by elementary means, without using the mathematical techniques common in tomography (c’mon, what can be difficult about $\sin x$, after all?), stalled for a while. We have succeeded eventually, albeit as we hope to persuade the reader, there are still things that are not clear about this whole business. Along the way, while trying to understand the identities, an interesting theorem of complex analysis in several variables was discovered in the joint work [1] of one of the authors and his student with L. Ehrenpreis. We dedicate this article to the memory of Professor Leon Ehrenpreis, who passed away in August 2010 (see [11, 33]).

In the next Section 2 we describe briefly the medical imaging origin of the results we will be discussing. Section 3 contains the formulation of the identities, which are then proved in the next three Sections. The paper ends with the remarks section, where in particular some

Research of P. K. was supported in part by the NSF DMS Grant 0604778 and IAMCS.

open questions and relations to several complex variables theory are described.

2. A BRIEF STORY ABOUT MEDICAL IMAGING, RADON TRANSFORM, AND ALL THAT

In tomographic medical imaging one sends through the patient's body some kind of radiation/waves (X -rays, ultrasound waves, electromagnetic waves, etc.), measures the transmitted and/or reflected signals, and tries to recover from this data the internal distribution of some important for medical diagnostics internal parameter of the body (e.g., density, electrical conductivity, blood oxygenation, etc.).

The reader must have heard of at least some tomographic procedures (the best-known ones are the X -ray CT scan and MRI). However, not everyone might realize the heavy involvement of sophisticated, beautiful, and challenging mathematics in creating tomographic images. In spite of its 40 years history, computed tomography still produces a steady stream of exciting mathematical problems (see, e.g., [3, 5, 7, 9, 10, 12–18, 21, 23–25, 28, 29, 34, 36–40] for description and mathematics of established and newly developing imaging modalities).

In the oldest and best established modality, X -ray CT, one deals with the problem of recovering a function $f(x)$ that is, roughly, the tissue density at the internal point x . Since in many cases different tissues have different densities, this function carries medically important information about the interior of the patient's body. If $f(x)$ is found, its density plot provides a picture, **tomogram** that shows to a doctor how an internal slice of the patient's body “looks like.” Mathematically speaking, the data obtained by the X -ray CT scanner provides the practitioner with line integrals of the sought for function f (see, e.g., [7, 10, 17, 23, 24] for detailed explanation). In other words, the data one measures is the so-called **2D Radon transform** that takes function $f(x)$ on the plane¹ and integrates it over all lines L :

$$f(x) \mapsto Rf(L) = \int_L f(x) dl.$$

We will parametrize lines by their **normal coordinates** (ω, s) , where ω is a unit vector normal to L and s is the (signed) distance from the origin to L . Then the **normal equation** of the line L has the form

$$x \cdot \omega = s,$$

¹We will not be precise about conditions on functions we deal with here. It is safe to assume that they are “very nice”: differentiable as many times as you want and vanishing at large distances.

where $x \cdot \omega$ is the standard inner product. Thus, the Radon transform Rf of a function f can be written as follows:

$$(1) \quad Rf(\omega, s) = g(\omega, s) := \int_{x \cdot \omega = s} f(x) dl = \int_{-\infty}^{\infty} f(s\omega + t\omega^\perp) dt.$$

Here $\omega^\perp = (\omega_2, -\omega_1)$ is the rotated 90° vector ω .

The Radon transform, besides being extremely useful in tomography, arises in a large number of other areas, e.g. in PDEs, group representation theory, etc., and thus has been the topic of thorough mathematical study [5, 12–15, 24, 25, 28, 29].

We are interested here in one feature particular for the Radon transform: its range as of a linear operator between natural function spaces is rather small, i.e. of infinite co-dimension. In other words, there are infinitely many conditions a function $g(\omega, s)$ must satisfy to be the Radon transform of a function f . Let us discuss them briefly.

One such condition is **evenness**:

$$(2) \quad g(-\omega, -s) = g(\omega, s) \text{ for all } \omega \in SS, s \in \mathbb{R},$$

where SS is the unit circle. This reflects the fact that equations $x \cdot \omega = s$ and $x \cdot (-\omega) = -s$ describe the same line.

More interesting are the so-called moment conditions. To formulate them, we assume that f has a compact support (in fact, decay faster than any power of $|x|$ at infinity suffices). Then for any integer $k \geq 0$ one can define **the k -th moment of the data** $g(\omega, s)$:

$$(3) \quad G_k(\omega) := \int_{-\infty}^{\infty} s^k g(\omega, s) ds,$$

which is a function on the circle SS .

Now the **moment conditions** say that

$$(4) \quad \begin{array}{l} \text{For any integer } k \geq 0, \text{ the moment } G_k(\omega) \\ \text{can be extended from } SS \text{ to the whole plane} \\ \text{as a homogeneous polynomial of degree } k. \end{array}$$

Although it might not be immediately obvious how one discovers the moment conditions, as soon as they are formulated, it is straightforward to check their necessity. Indeed, substituting into (3) the definition of

the Radon transform $g = Rf$, one gets

$$G_k(\omega) := \int_{-\infty}^{\infty} s^k g(\omega, s) ds = \int_{-\infty}^{\infty} s^k \int_{x \cdot \omega = s} f(x) dl ds = \int_{\mathbb{R}^2} (x \cdot \omega)^k f(x) dx.$$

The last expression is clearly a homogeneous polynomial of degree k of ω (since $x \cdot \omega$ is a homogeneous linear function of ω) and thus we have established these so-called “range conditions”.

Proving that there are no other range conditions besides evenness and moment ones, is a different story, which takes much more work (see, e.g. [5, 13, 14, 24, 25]).

Why would one be interested in these conditions? They are the mandatory relations that ideal data collected from a tomographic device must satisfy. Well, the measured data are never ideal, and so they deviate from these conditions. Thus, knowing the range conditions might be helpful in detecting and correcting some measurement errors. They are of help in other circumstances as well, for instance in completing some missing data (e.g., see [22, 24, 25, 30] and references therein for details). And they surely play an important role in most of analysis of the Radon transform as an operator.

Quite a few years ago, the authors worked [19, 20] on finding the range conditions for some special (weighted) Radon-type transform arising in another popular medical imaging method, so-called SPECT (Single Photon Emission Computed Tomography) [9, 16, 18, 21, 24]. This is the so-called **exponential Radon transform**, which is defined as follows:

$$(5) \quad R_\mu f(\omega, s) = g(\omega, s) := \int_{-\infty}^{\infty} f(s\omega + t\omega^\perp) e^{\mu t} dt.$$

Here $\mu > 0$ is the **attenuation coefficient**.

When we found a set of range conditions² for R_μ and then proved that we had not missed any, we thought that checking their necessity (as long as we already knew them) should be a piece of cake, just like in the example of the usual Radon transform above. You plug your transform $R_\mu f$ into the conditions and should be able to “see” immediately that they are satisfied. Well, when we did this, we saw an infinite series of identities for $\sin x$ (yes, the usual sine!) that we could not recognize and did not know why they should have been true. Here they are:

²We do not provide these conditions here, since they are rather technical and do not make what follows easier. The interested reader can find them in [18–20].

For any odd natural number n , the following identity holds:

$$(6) \quad \sum_{k=0}^n \binom{n}{k} \left(\frac{d}{dx} - \sin x \right) \circ \left(\frac{d}{dx} - \sin x + i \right) \circ \cdots \circ \left(\frac{d}{dx} - \sin x - (k-1)i \right) (\sin x)^{n-k} \equiv 0.$$

Here i is the imaginary unit, $\binom{n}{k}$ is the binomial coefficient “ n choose k ,” the expressions in parentheses are considered as differential operators, and \circ denotes their composition. For instance,

$$\left(\frac{d}{dx} - \sin x \right) u(x) = \frac{du}{dx} - \sin x u(x).$$

Although we have provided the necessary proofs and moved on to doing other things, the true meaning of these identities has kept us puzzling for all these years.

In this paper we present the formulation and elementary algebraic derivation of these identities (not only for $\sin x$, but even for some more “elementary” functions such as linear and exponential ones).

We would be happy if someone could shed some more light onto the meaning and possible generalizations of these identities.

3. ALGEBRAIC FORMULATION OF THE IDENTITIES

The identities (6) can be formulated and proven in a rather general algebraic setting. Namely, let A be a commutative algebra with cancellation property and with a differentiation³ D over a field K . An element $u \in A$ can be considered an analog of the sine function if it satisfies the equation $D^2u = \lambda^2u$ for some $\lambda \in K$. For instance, in the case when $u = \sin x$, we are talking about the algebra $C^\infty(\mathbb{R})$ over complex numbers of all smooth functions on the variable x , with differentiation $D = d/dx$ and $\lambda = i$.

In what follows, the reader will not lose anything thinking, instead of the general algebraic situation, of smooth functions $f(x)$ with the usual derivative $D = d/dx$. However, the general algebraic notations (A, D, λ) lead to somewhat easier to read formulas, so we will stick with them most of the time.

We are interested in solutions of simple differential equations

$$(7) \quad Du = \lambda u$$

and

$$(8) \quad D^2u = \lambda^2u,$$

³I.e., $D(uv) = D(u)v + uD(v)$.

where $\lambda \in K$. Thus, in the example of the function algebra with usual differentiation, solutions of equation (7) for $\lambda = 0$ are constants, while for non zero values of λ they give us exponential functions. Analogously, solutions of (8) for $\lambda = 0$ are linear functions, while for non zero values of λ they in particular give us trigonometric and hyperbolic sine and cosine.

We now formulate the main results.

Theorem 1. *Any solution u of the first-order equation $Du = \lambda u$ satisfies for any natural n the following identity:*

$$(9) \quad \sum_{k=0}^n \binom{n}{k} (D-u) \circ (D-u+\lambda) \circ \dots \circ (D-u+(k-1)\lambda) u^{n-k} = 0.$$

Theorem 2. *Any solution u of the second-order equation $D^2u = \lambda^2u$ satisfies for any **odd** natural n the identity (9).*

The expression u^{n-k} at the very right is just the $(n-k)$ th power of u . Expressions like $(D-u)$ are considered as **operators** on the algebra A . For instance, $(D-u)$ applied to an element $f \in A$ produces $(D-u)f = Df - uf$. The little circles \circ mean **composition** of these operators from right to left. For instance,

$$\begin{aligned} (D-u) \circ (D-u+\lambda)f &= \left(\frac{d}{dx} - u(x)\right) \left(\frac{df}{dx} - u(x)f(x) + \lambda f(x)\right) \\ &= \frac{d^2f}{dx^2} - \frac{d(uf)}{dx} + \lambda \frac{df}{dx} - u(x) \frac{df}{dx} + u^2(x)f(x) - \lambda u(x)f(x). \end{aligned}$$

So, the expression in (9) directs us to raise u to the power $(n-k)$, then apply to it the operator $(D-u+(k-1)\lambda)$, then to apply to the resulting function the operator $(D-u+(k-2)\lambda)$, etc. To avoid confusion in the case of $k=0$, it might be easier to think that the composition in (9) has k operator factors. In other words, the term for $k=0$ does not have any of these factors and thus is just u^n .

An important thing to notice is that, as Calculus teaches us, the operations D and u (differentiation and multiplication by u) do not commute, i.e. the result of their composition depends on its order. If this were not so, the identities (9) would become trivial, as we will see a little bit later.

Since solutions of the equations (7)-(8) are very simple functions (constants and exponential, linear, trigonometric or hyperbolic functions), let us look at some examples to see whether the identities (9) might be trivial, or at least look familiar.

Take (7) for $\lambda = 0$, so the solution u is just a constant C . Then (9) looks as follows:

$$\sum_{k=0}^n \binom{n}{k} (D - C)^k C^{n-k} = 0.$$

Since D acting on any constant gives zero result, this identity boils down to

$$\sum_{k=0}^n \binom{n}{k} (-C)^k C^{n-k} = (-C + C)^n = 0.$$

This is indeed easy and proves the particular case of Theorem 1 for $\lambda = 0$.

Consider now the particular case of (7) when $\lambda = 1$ and $u = e^x$. Then (9) reduces to

$$\sum_{k=0}^n \binom{n}{k} \left(\frac{d}{dx} - e^x \right) \circ \left(\frac{d}{dx} - e^x + 1 \right) \circ \dots \circ \left(\frac{d}{dx} - e^x + (k-1) \right) e^{(n-k)x} = 0.$$

Hmmm... Does not look familiar? Neither it did to us. Well, we will not prove this particular identity, since we are going to prove the more general Theorems 1 and 2.

Let us look at an example of a solution of the second-order equation $D^2u = \lambda^2 u$, say take $u = \sin x$ and $\lambda = i$. Then the identity (9) for $n = 3$ becomes (using more familiar $\frac{d}{dx}$ instead of D)

(10)

$$\begin{aligned} \sin^3 x + 3\left(\frac{d}{dx} - \sin x\right) \sin^2 x + 3\left(\frac{d}{dx} - \sin x\right) \circ \left(\frac{d}{dx} - \sin x + i\right) \sin x \\ + \left(\frac{d}{dx} - \sin x\right) \circ \left(\frac{d}{dx} - \sin x + i\right) \circ \left(\frac{d}{dx} - \sin x + 2i\right) 1 = 0. \end{aligned}$$

Obvious, isn't it? Well, do not despair if you do not see it right away, we don't either.

As you will see, the direct algebraic proofs of Theorems 1 and 2 (at least the proofs that we know) are different when $\lambda = 0$ and when $\lambda \neq 0$. Certainly, it is not hard to derive the identities for $\lambda = 0$ from the ones for $\lambda \neq 0$ by taking the limit when $\lambda \rightarrow 0$. We, however, would like to see direct algebraic proofs of the identities for both cases $\lambda = 0$ and $\lambda \neq 0$. Such proofs are provided below.

4. PROOF OF THEOREM 1

We have already proven it above for $\lambda = 0$. So, let us assume that $\lambda \neq 0$.

Suppose that u satisfies the first-order equation $Du = \lambda u$. We can apply a **gauge transform** to simplify our identities. Namely, we will use the easy to check identity

$$(11) \quad u^{-m}(D - u)u^m = D - u + m\lambda.$$

We can now use (11) in each factor of (9) to rewrite it as follows:

$$(12) \quad \left(\sum_{k=0}^n \binom{n}{k} [(D - u) \circ u^{-1}]^k \right) u^n = 0.$$

Here we notice that we deal with an operator binomial $\sum_{k=0}^n \binom{n}{k} A^k$, where $A = (D - u) \circ u^{-1}$. Since only one operator is involved, no non-commutativity arises, and thus the usual binomial formula works. This reduces identity (9) to

$$(13) \quad [(D - u) \circ u^{-1} + 1]^n u^n = (D \circ u^{-1})^n u^n = 0.$$

Now an immediate calculation using $Du = \lambda u$ shows that $(D \circ u^{-1})^n u^n = 0$ holds.

Remark 3. *A careful reader might object to our calculations in this section that involve negative powers of u , as well as positive ones, especially in the abstract setting of a commutative differential algebra A with cancelation. However, this algebraic problem can be overcome (see [19]).*

5. PROOF OF THEOREM 2 FOR $\lambda = 0$

We assume now that $\lambda = 0$ and thus the function u satisfies the equation $D^2u = 0$. In this case the identities (9) that we intend to prove simplify to

$$(14) \quad \sum_{k=0}^n \binom{n}{k} (D - u)^k u^{n-k} = 0$$

for any odd natural n .

Remark 4. *This identity fails for $n = 2$.*

One can understand (14) also as follows:

$$(15) \quad \left(\sum_{k=0}^n \binom{n}{k} (D - u)^k \circ u^{n-k} \right) 1 = 0.$$

The expression in the parentheses resembles the familiar binomial expression

$$\sum_{k=0}^n \binom{n}{k} a^k b^{n-k},$$

where $a = D - u$ and $b = u$. If the operators commuted, then the whole thing would have been equal to $(a + b)^n = (D - u + u)^n 1 = D^n 1 = \frac{d^n 1}{dx^n} = 0$. Done!

Hold on, not so fast! First of all, since D and u do not commute, the usual binomial formula does not apply. Besides, it looks like we obtained the identity (9) for **all** natural n , which is incorrect, as we saw in Remark 4. So, let us take a closer look.

If $\lambda = 0$, then function u is linear. Let us concentrate on the case when $u(x) = x$.

Let us first rewrite our identity (15) for the case when $u = x$:

$$(16) \quad \sum_{k=0}^n \binom{n}{k} (D - x)^k x^{n-k} = 0.$$

We will now use another popular **gauge transformation** trick that will enable us to rewrite (16) in a slightly different form. It is based on the following simple identity:

$$(17) \quad (D - x)^k \left(e^{\frac{x^2}{2}} f(x) \right) = e^{\frac{x^2}{2}} D^k f(x).$$

Thus, commutation with the function $e^{\frac{x^2}{2}}$ kills the term $-x$ added to the derivative and $(D - x)^k$ becomes D^k :

$$e^{-\frac{x^2}{2}} \circ (D - x)^k \circ e^{\frac{x^2}{2}} = D^k.$$

This implies the following

Lemma 5. *The identities (16) are equivalent to*

$$(18) \quad \left(\sum_{k=0}^n \binom{n}{k} D^k \circ x^{n-k} \right) e^{-\frac{x^2}{2}} = 0.$$

So, now the proof of Theorem 2 for $\lambda = 0$ boils down to proving (18) for odd n . Notice that if D and x commuted, then the expression in parentheses in (18) would become just $(D + x)^n$, and thus, due to the easily verified equality

$$(19) \quad (D + x) e^{-\frac{x^2}{2}} = 0,$$

we would immediately obtain the validity of (18). However, life is not so easy and thus the operators do not commute. (Besides, as we already know, (18) is guaranteed for odd values of n only).

Based on (18), let us introduce the differential operators

$$P_n(D, x) = \sum_{k=0}^n \binom{n}{k} D^k \circ x^{n-k}.$$

The punch line is in the following statement:

Lemma 6. *The following recurrence relation holds:*

$$(20) \quad P_{n+2}(D, x) = P_{n+1}(D, x) \circ (D + x) + (n+1)P_n(D, x).$$

Proof of the Lemma. We will compute $P_{n+1}(D, x) \circ (D + x)$ and show that it coincides with $P_{n+2}(D, x) - (n+1)P_n(D, x)$. In order to do so, we will use the easy to verify commutation relation

$$D \circ x^m = x^m \circ D + mx^{m-1}$$

and its consequence

$$(21) \quad x^m \circ D = D \circ x^m - mx^{m-1}.$$

Now, take a deep breath, and ...

$$(22) \quad \begin{aligned} P_{n+1}(D, x) \circ (D + x) &= \left(\sum_{k=0}^{n+1} \binom{n+1}{k} D^k \circ x^{n+1-k} \right) \circ (D + x) \\ &= \sum_{k=0}^{n+1} \binom{n+1}{k} D^k \circ x^{n+1-k} \circ D + \sum_{k=0}^{n+1} \binom{n+1}{k} D^k \circ x^{n+2-k}. \end{aligned}$$

Let us now use (21) to commute x^{n+1-k} with D in the first sum to get

$$(23) \quad \begin{aligned} &\sum_{k=0}^{n+1} \binom{n+1}{k} D^{k+1} \circ x^{n+1-k} + \sum_{k=0}^{n+1} \binom{n+1}{k} D^k \circ x^{n+2-k} \\ &- \sum_{k=0}^{n+1} (n+1-k) \binom{n+1}{k} D^k \circ x^{n-k}. \end{aligned}$$

The first two sums can be rewritten as

$$(24) \quad \begin{aligned} &\sum_{k=1}^{n+2} \binom{(n+2)-1}{k-1} D^{k+1} \circ x^{(n+2)-k} + \sum_{k=0}^{(n+2)-1} \binom{(n+2)-1}{k} D^k \circ x^{(n+2)-k} \\ &= \sum_{k=0}^{n+2} \binom{n+2}{k} D^k \circ x^{(n+2)-k} = P_{n+2}(D, x). \end{aligned}$$

We used here the standard property of the Pascal triangle:

$$\binom{(n+2)-1}{k-1} + \binom{(n+2)-1}{k} = \binom{n+2}{k}.$$

It only remains to handle the remainder

$$\sum_{k=0}^{n+1} (n+1-k) \binom{n+1}{k} D^k \circ x^{n-k}.$$

This expression is equal to $(n+1)P_n(D, x)$, proving which only requires the knowledge of what binomial coefficients are. This finishes the proof of the Lemma. \square

Corollary 7. *For any odd natural n , the following factorization holds with some operator $Q(D, x)$:*

$$(25) \quad P_n = Q \circ (D + x).$$

Proof. Let us prove this by induction. When $n = 1$, we have $P_1 = D + x$, and the statement is obvious. Assume that we have proven it for some odd n , i.e. $P_n = Q \circ (D + x)$. Then the previous Lemma implies that $P_{n+2} = (P_{n+1} + (n+1)Q) \circ (D + x)$. \square

Remark 8. *The factorization $P_n = Q \circ (D + x)$ fails for $n = 2$.*

Now the proof of Theorem 2 for $\lambda = 0$ is immediate. Indeed, for any odd n , the left hand side in the identity (18) in question becomes $P_n e^{-\frac{x^2}{2}} = Q \circ (D + x) e^{-\frac{x^2}{2}}$. Applying (19), we conclude that this is zero. This finishes the proof of the Theorem for $\lambda = 0$ and $u = x$.

The proof provided for $u = x$ can be easily generalized to an arbitrary linear function $u = ax + b$. Alternatively, one can derive the result for any linear function from the one for $u = x$. \square

6. PROOF OF THEOREM 2 FOR $\lambda \neq 0$

Let us now outline the steps of the proof of Theorem 2 in case $\lambda \neq 0$.

Step 1. If u satisfies the first-order equation $Du = \lambda u$ (and thus certainly $D^2u = \lambda^2 u$ as well), the statement follows from Theorem 1.

Step 2. Suppose $Du = -\lambda u$. Then (9) is true for odd n , because one can easily check that “symmetric” terms in the sum that correspond to k and $n - k$ cancel out.

Step 3. Let us now suppose that u is any solution of $D^2u = \lambda^2 u$. If our identities were linear with respect to u (which they are not), then the cases when $Du = \pm \lambda u$ discussed in the two previous steps would suffice, since any solution could be expanded into a sum of these two. Nonlinearity seems to destroy this idea. However, having nothing better in mind, let us still try. Thus, $u = v + w$, where $Dv = -\lambda v$

and $Dw = \lambda w$ (notice that the assumption $\lambda \neq 0$ is critical for the possibility of such decomposition)⁴.

Step 4. Let us use the same commutation trick (11)

$$u^{-m}(D - u)u^m = D - u + m\lambda$$

as before, but using commutation with w rather than u . Then the left-hand side of (9) can be written as

$$(26) \quad \begin{aligned} & \sum_{k=0}^n \binom{n}{k} (D - v - w) \circ \dots \circ (D - v - w + (k-1)\lambda)(v + w)^{n-k} \\ &= \left[\sum_{k=0}^n \binom{n}{k} [(D - v) \circ w^{-1} - 1]^k (vw^{-1} + 1)^{n-k} \right] w^n. \end{aligned}$$

Step 5. Let us introduce the following operator notations: $A = (D - v) \circ w^{-1}$ and $B = vw^{-1}$. Then the last sum becomes

$$\sum \binom{n}{k} (A - 1)^k (B + 1)^{n-k}.$$

If the operators A and B commuted, then according to the binomial formula this would boil down to $(A+B)^n$ and thus also to $\sum \binom{n}{k} A^k B^{n-k}$. The interesting thing is that the latter conclusion holds even without commutativity:

Lemma 9. *For any two operators A, B the following equality holds:*

$$\sum \binom{n}{k} (A - 1)^k (B + 1)^{n-k} = \sum \binom{n}{k} A^k B^{n-k}.$$

Indeed, this equality does not require changing the order of the factors. Thus, if it holds for commuting operators, then it does for non-commuting ones as well.

How can this help us with the proof of the theorem? It allows us to drop the terms ∓ 1 in (26) to get

$$\left[\sum_{k=0}^n \binom{n}{k} [(D - v) \circ w^{-1}]^k (vw^{-1})^{n-k} \right] w^n.$$

Now one can reverse Step 4 (undoing the commutations with powers of w we have done) and rewrite (26) as

$$\sum_{k=0}^n \binom{n}{k} (D - v) \circ \dots \circ (D - v + (k-1)\lambda) v^{n-k},$$

⁴This is just standard exponential representation of solutions studied in ODEs (or, if you will, the representation of $\sin x$ as combination of $e^{\pm ix}$).

which is equal to zero due to the Step 2, since v solves the first-order equation $Dv = -\lambda v$. This proves Theorem 2 for $\lambda \neq 0$. \square

7. REMARKS AND GENERALIZATIONS

- For the readers with a taste for generalizations, we can mention that the identities (9) hold in a much more general situation. Namely, u can be assumed to be an element of a commutative algebra with cancelation and with differentiation D over a field Λ . Then λ should be an element of the field Λ [19].
- Generalizations of some of the identities we discussed are available:

Theorem 10. *If u satisfies $D^2u = 0$, then the identity*

$$(27) \quad \sum_{k=0}^n \binom{n}{k} (D - u)^k \circ D^m u^{n-k} = 0$$

holds for all odd integer $n \geq 1$ and even $m \geq 0$.

- Several integral geometry and tomography experts devoted their time and effort to trying to understand better the meaning of these strange range conditions. This is also what one of the authors set out to do with V. Aguilar and L. Ehrenpreis. Surprisingly, as the result, the identities (9) were related [1] to the so-called separate analyticity theorems (Hartogs-Bernstein theorems) in several complex variables. For instance, the following amazing theorem is essentially equivalent to these identities [1, 26, 27]:

Theorem 11. *Let Ω be a disk in \mathbb{R}^2 and $f(x)$ be a function in the exterior of Ω . Suppose that when restricted to any tangent line L to $\partial\Omega$, the function $f|_L$, as a function of one real variable extends to an entire function on the complexification of L . Then f , as a function on $\mathbb{R}^2 \setminus \Omega$ extends to an entire function on \mathbb{C}^2 .*

Well, this fact also did not look obvious. Analyticity of f in a complex neighborhood of $\mathbb{R}^2 \setminus D$ follows from the old (and not well known) separate analyticity theorem by S. Bernstein (see [2]), however this theorem cannot produce statement about f being an entire function. Thus, since proving the above theorem, a couple of things about it kept bothering us for several years. First of all, this is a several complex variables fact, while our proof did not look like a SCV argument at all. Is there a truly complex analysis proof? Another, related, question is whether such a theorem can be proven for a different

convex body instead of a disk Ω ? A SCV proof was later provided in [26, 27], although it was rather complicated and was not generalizable (at least, easily) to other convex curves. Leon Ehrenpreis has worked out some other examples of convex algebraic curves (unpublished), but general picture remained unclear. Finally, A. Tumanov presented recently [35] a beautiful short proof based on attachment of analytic disks (where Tumanov is a great expert), which works for any strictly convex body Ω with a mild conditions on the smoothness of its boundary.

More discussion of the relations of the range conditions with complex analysis can be found for instance in [6, 18–20].

- It would be interesting to find an algebraic proof of the identities that would work simultaneously for $\lambda = 0$ as well as for $\lambda \neq 0$.
- It is clear that the identities discussed must be related to special function theory and group representations. It would be interesting to understand such relations.
- The formulations of the Theorems 1 and 2 lead to the natural question: what can be said for solutions of the equation of 3rd order $D^3u = \lambda^3u$ and higher? A natural guess would be that the same identities hold, but only for an arithmetic sequence of numbers n with the difference equal to 3. It is hard to compute these expressions by hand even for small values of n , say for $n = 4$. Ms. E. Rodriguez, a former Masters student of P. K., has used the Maple symbolic algebra system to check this conjecture. The result is negative, the natural conjecture fails for equations of third order [31]. So, what (if anything) happens to solutions of higher order differential equations $D^m u = \lambda^m u$? Are they deprived of any such identities? We do not know the answer to this question.

ACKNOWLEDGMENTS

The authors express their gratitude to S. Fulling, D. Hensley, Z. Sunik and G. Tee for useful comments.

REFERENCES

- [1] V. Aguilar, L. Ehrenpreis, and P. Kuchment, Range conditions for the exponential Radon transform, *J. d'Analyse Mathématique*, 68(1996), 1–13.
- [2] N. I. Akhiezer and L. I. Ronkin, On separately analytic functions of several variables and theorems on “the edge of the wedge”, *Russian Math. Surveys* **28** (1973), No. 3, 27–42.

- [3] G. Bal, D. Finch, P. Kuchment, P. Stefanov, G. Uhlmann (Editors), *Tomography and Inverse Transport Theory*, AMS, to appear.
- [4] T.F. Budinger, G.T. Gullberg, and R.H. Huseman, Emission computed tomography, pp. 147-246 in [16].
- [5] L. Ehrenpreis, *The Universality of the Radon Transform*, Oxford Univ. Press 2003.
- [6] L. Ehrenpreis, P. Kuchment, and A. Panchenko, The exponential X-ray transform and Fritz John's equation. I. Range description, in *Analysis, Geometry, Number Theory: the Mathematics of Leon Ehrenpreis (Philadelphia, PA, 1998)*, 173–188, Contemporary Math., 251, Amer. Math. Soc., Providence, RI, 2000.
- [7] Charles L. Epstein, *Mathematics of Medical Imaging*, Prentice-Hall 2003.
- [8] H. Farkas, T. Kawai, P. Kuchment, T. E. Quinto, S. Sternberg, D. Struppa, and B. A. Taylor, Remembering Leon Ehrenpreis: 1930-2010, Notices of the Amer. Math. Soc. **58** (2011), no. 5, 674–681.
- [9] A. S. Fokas and L.-Y. Sung, Generalized Fourier transforms, their nonlinearization and the imaging of the brain, Notices of the AMS **52** (2005), 1176-1190.
- [10] Timothy G. Feeman, *The Mathematics of Medical Imaging: A Beginners Guide* (Springer Undergraduate Texts in Mathematics and Technology), Springer Verlag 2009.
- [11] H. Farkas, T. Kawai, P. Kuchment, T. E. Quinto, S. Sternberg, D. Struppa, and B. A. Taylor, Remembering Leon Ehrenpreis: 1930-2010, Notices of the Amer. Math. Soc. **58** (2011), no. 5, 674–681.
- [12] I. Gelfand, M. Graev M. and N. Vilenknin. *Generalized Functions, v. 5: Integral Geometry and Representation Theory*, Acad. Press. 1965.
- [13] I. Gelfand, S. Gindikin, and M. Graev, *Selected Topics in Integral Geometry*, Transl. Math. Monogr. v. 220, Amer. Math. Soc., Providence RI, 2003.
- [14] S. Helgason, *The Radon Transform*, Birkhäuser, Basel 1980.
- [15] S. Helgason. *Integral Geometry and Radon Transforms*, Springer, NY, 2010.
- [16] G. Herman (Ed.), *Image Reconstruction from Projections*, Topics in Applied Physics, v. 32, Springer Verlag, Berlin, New York 1979.
- [17] Avinash C. Kak and Malcolm Slaney, *Principles of Computerized Tomographic Imaging*, Soc. for Industrial and Applied Math. 2001.
- [18] P. Kuchment. Generalized Transforms of Radon Type and Their Applications. in [28], 2006, pp. 67–91.
- [19] P. Kuchment, S. Lvin, Paley-Wiener theorem for exponential Radon transform, *Acta Applicandae Mathematicae* **18** (1990) 251-260.
- [20] P. Kuchment and S. Lvin, The range of the exponential Radon transform, Soviet Math. Dokl. 42(1991), no.1, 183–184.
- [21] P. Kuchment and E. T. Quinto, Some problems of integral geometry arising in tomography, chapter XI in the book [5].
- [22] S. Lvin, Data correction and restoration in emission tomography, *Mathematics of Tomography, Impedance Imaging and Integral Geometry – Lectures in Applied Mathematics* **30** (1994) 149-155.
- [23] *Mathematics and Physics of Emerging Biomedical Imaging*, The National Academies Press 1996. Available online at http://www.nap.edu/catalog.php?record_id=5066#toc.

- [24] F. Natterer, *The mathematics of computerized tomography*, Wiley, New York, 1986. Reprint by Soc. Industrial and Applied Math. 2001.
- [25] F. Natterer and F. Wübbeling, *Mathematical Methods in Image Reconstruction*, Monographs on Mathematical Modeling and Computation v. 5, SIAM, Philadelphia, PA 2001.
- [26] O. Öktem, Comparing range characterizations of the exponential Radon transform, Research reports in Math., no.17, 1996, Dept. Math., Stockholm Univ., Sweden.
- [27] O. Öktem, Extension of separately analytic functions and applications to range characterization of the exponential Radon transform, in *Complex Analysis and Applications (Warsaw, 1997)*, Ann. Polon. Math. 70(1998), 195–213.
- [28] G. Olafsson and E. T. Quinto (Editors). *The Radon Transform, Inverse Problems, and Tomography. American Mathematical Society Short Course* January 3-4, 2005, Atlanta, Georgia, Proc. Symp. Appl. Math. 63, AMS, RI, 2006.
- [29] V. P. Palamodov, *Reconstructive Integral Geometry*, Birkhäuser, Basel 2004.
- [30] I. Ponomarev, Correction of emission tomography data. Effects of detector displacement and non-constant sensitivity, Inverse Problems, 10(1995) 1–8.
- [31] E. Rodriguez, unpublished.
- [32] O. Scherzer (Editor), *Handbook of Mathematical Methods in Imaging*, Springer Verlag 2010
- [33] D. Struppa (Editor), *The Mathematical Legacy of Leon Ehrenpreis*, to appear.
- [34] V. V. Tuchin (Editor), *Handbook of Optical Biomedical Diagnostics*, SPIE, WA 2002.
- [35] A. Tumanov, Analytic continuation from a family of lines, J. Anal. Math. **105** (2008), 391–396.
- [36] G. Uhlmann (Editor), *Inside out: Inverse Problems and Applications*, MSRI Publ., Cambridge University Press, 2003.
- [37] G. Uhlmann (Editor), *Inside out: Inverse Problems and Applications*, Vol. 2, MSRI Publ., to appear.
- [38] T. Vo-Dinh (Editor), *Biomedical Photonics Handbook*, edited by CRC, 2003.
- [39] L. V. Wang (Editor), *Photoacoustic imaging and spectroscopy*, CRC Press 2009
- [40] L. Wang and H. Wu, *Biomedical Optics*, Wiley 2007

MATHEMATICS DEPARTMENT, TEXAS A& M UNIVERSITY, COLLEGE STATION,
TX 77843-3368

E-mail address: kuchment@math.tamu.edu

URL: <http://www.math.tamu.edu/~kuchment>

DEPARTMENT OF MATHEMATICS AND STATISTICS, UNIVERSITY OF MAINE,
ORONO, ME 04469-5257

E-mail address: Lvin@math.umaine.edu